Discussion 17 Worksheet

Some past exam problems

Date: 11/8/2021

MATH 53 Multivariable Calculus

1 Computing Curl and Divergence

For each of the following vector fields $\vec{F}$, compute its curl and divergence. State whether each vector field is irrotational, incompressible, or neither.

1. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$
2. $\vec{F} = \langle y^2, z^3, x^4 \rangle$
3. $\vec{F} = \langle y^2 x, e^z, z^2 \rangle$
4. $\vec{F} = \nabla f$, where $f(x, y, z) = 2xye^{yz}$

2 Divergence and Curl Identities

Let $\vec{F} = \langle P, Q, R \rangle$ and $\vec{G} = \langle P', Q', R' \rangle$ be vector fields on $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be functions on $\mathbb{R}^3$. Assume all of these are infinitely differentiable. Prove each of the following vector identities.

1. $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$
2. $\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G}$
3. $\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + \vec{F} \cdot (\nabla f)$
4. $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$
5. $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$
6. $\nabla \cdot (\nabla f \times \nabla g) = 0$
7. $\nabla \times (\nabla \times \vec{F})) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

3 Challenge

Suppose you are given a pair of (infinitely differentiable) vector fields $\vec{E}$ and $\vec{B}$ in $\mathbb{R}^3$ in $\mathbb{R}^3$, and consider each vector field as additionally varying with respect to a variable t (in addition to the variables x, y, and z for $\mathbb{R}^3$). Suppose furthermore that these vector fields satisfy the "Maxwell equations in vacuum:"

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

for some constant $c^2 > 0$. Prove that these vector fields satisfy the "wave equations"

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad\qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}.$$

Here $\nabla^2 \vec{E}$ is the vector Laplacian

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2},$$

and $\nabla^2 \vec{B}$ is defined similarly (with $\vec{E}$ replaced by $\vec{B}$).

By completing this exercise, you are showing that the fundamental laws of electrodynamics suggest the possibility of electromagnetic waves, i.e. light. *Fiat lux!*

4 True/False

Supply convincing reasoning for your answer. Assume all functions are infinitely differentiable unless stated otherwise.

(a) T F For every vector field $\vec{F}$ on $\mathbb{R}^3$, we have $\nabla \cdot (\nabla \times \vec{F}) = 0$.

(b) T F The divergence of a vector field is a scalar function, while the curl of a vector field is a vector field.

(c) T F Every vector field $\vec{F}$ on $\mathbb{R}^3$ arises as the curl of some vector field $\vec{G}$.

(d) T F Every function $f : \mathbb{R}^3 \to \mathbb{R}$ arises as the divergence of some vector field $\vec{F}$.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.